The Qualitative Paradox of Non-Conglomerability*

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Abstract

A probability function is non-conglomerable just in case there is some proposition E and partition π of the space of possible outcomes such that the probability of E conditional on any member of π is bounded by two values yet the unconditional probability of E is not bounded by those values. The paradox of non-conglomerability is the counterintuitive—and controversial—claim that a rational agent's subjective probability function can be non-conglomerable. In this paper, I present a qualitative analogue of the paradox. I show that, under antecedently plausible assumptions, an analogue of the paradox arises for rational comparative confidence. As I show, the qualitative paradox raises its own distinctive set of philosophical issues.

1 Introduction

Arntzenius et al. (2004) introduce the intuitively odd feature of certain probability functions known as 'non-conglomerability' as follows:

> Suppose that conditional on its being cold tomorrow, you are confident that it will be sunny. Suppose further that conditional on its *not* being cold tomorrow, you are *also* confident that it will be sunny. It would be odd indeed if you in addition were confident that it *was not* going to be sunny tomorrow. The odd feature your probability function has in this case is *non-conglomerability*. (p. 274; emphasis in original)

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More generally, a probability function P is non-conglomerable just in case there is some proposition E and partition π of the space of possible outcomes such that the probability of E conditional on any member of π is bounded by two values yet the unconditional probability of E is not bounded by those values.[1] In the above example:

- $E = \mathit{Sunny}$,
- $\pi = \{\mathit{Cold},\ \mathit{Not\ Cold}\}$,
- $0.5 < P(E|\mathit{Cold}) < 1$,
- $0.5 < P(E|\mathit{Not\ Cold}) < 1$, yet
- $P(E) < 0.5$.

Non-conglomerable probability functions are not just odd. Intuitively, it seems *irrational* for you to have a subjective probability function that is non-conglomerable. In the above example, either it will be cold tomorrow, or it will not. Further, given either disjunct, you are confident that it will be sunny. Thus, a simple application of the the deductive inference rule of "proof by cases" seems to entail that, unconditionally, you should be confident that it will be sunny. Yet you are confident that it will not be so. Hence, you seem to be in violation of deductive logic—as reputable a norm of rationality as any.[2]

Despite the intuitive oddness of non-conglomerable probability functions, it is uncontroversial that the axioms of probability theory permit their existence.[3] What *is* controversial is whether a rational agent's subjective probability function can ever be non-conglomerable. The claim that it can is the paradox of non-conglomerability.[4] Some have argued for the paradox and

[1] I provide a more precise definition in Sect. 2.

[2] I discuss the connection between non-conglomerability and "proof by cases" further in Sect. 4.3.

[3] More precisely, it is uncontroversial that the Kolmogorov axiomatization—when formulated to allow for infinite probability spaces—permits their existence. See de Finetti (1930), Kadane et al. (1986), and Arntzenius et al. (2004) for various examples of non-conglomerable probability functions. Jaynes (2003), who denies the legitimacy of infinite probability spaces, is a notable denier of the existence of non-conglomerable probability functions.

[4] So called by de Finetti (1972).

claimed that it is simply a counterintuitive fact about rationality.[5] Others have resisted the paradox by rejecting arguments that purportedly lead to it or by offering positive arguments against it.[6]

In this paper, I present a qualitative analogue of the paradox. I show that, under antecedently plausible assumptions, an analogue of non-conglomerability can arise for a rational agent's *comparative confidence relation*—where comparative confidence is the attitude of being at least as confident in one proposition as in another. Just as the more familiar quantitative paradox raises a number of philosophical issues, so I will show that the qualitative paradox raises its own distinctive set of philosophical issues. Over the course of the paper, I will show that the qualitative paradox has relevance to infinitesimals, monotone continuity, Jaynes (2003)'s probabilistic finitism, and the relation between numerical credence and comparative confidence. The plan is as follows.

In Sect. 2, I review the notion of a conglomerable probability function as well as de Finetti (1972)'s famous example of non-conglomerability. In Sect. 3, I lay out some constraints on rational comparative *conditional* confidence—namely, those of Koopman (1940a)—and define an analogous notion of conglomerability for an agent's comparative confidence relation. In Sect. 4, I show that de Finetti's non-conglomerable probability function can be reformulated as a rational comparative confidence relation that is non-conglomerable. In Sect. 5, I discuss the philosophical significance of this result. In Sect. 6, I discuss possible responses to the qualitative paradox. Finally, I close in Sect. 7 with open questions for future work.

2 Probabilistic non-conglomerability

To spell out the notion of a conglomerable probability function, we will need a few preliminary notions.

First, let $(\Omega, \mathcal{F}, P)$ be a conditional probability space. That is, let Ω be a set of possible outcomes, $\mathcal{F}$ a Boolean algebra on Ω, and $\mathcal{F}_0$ the set of non-empty elements of $\mathcal{F}$. Then, P is a real-valued function on $\mathcal{F} \times \mathcal{F}_0$ that satisfies:

1. For all $A \in \mathcal{F}, B \in \mathcal{F}_0$: $P(A|B) \geq 0$.

[5] See de Finetti (1972), Hill (1980), Kadane et al. (1986), and Arntzenius et al. (2004).

[6] See Jaynes (2003), Easwaran (2008, 2013a), Pruss (2012), and the responses of Dickey, Fraser, and Lindley in Hill (1980).

2. For all $A \in \mathcal{F}_0$: $P(A|A) = 1$.
3. (Finite Additivity.) For all $A, B \in \mathcal{F}$ and $C \in \mathcal{F}_0$ such that $A \cap B = \emptyset$:
$$P[(A \cup B)|C] = P(A|C) + P(B|C).^{7} \tag{1}$$
4. For all $A, B, C \in \mathcal{F}$ such that $(B \cap C) \in \mathcal{F}_0$:
$$P[(A \cap B)|C] = P[A|(B \cap C)]P(B|C).^{8} \tag{2}$$

Additionally, define unconditional probability via $P(A) =_{df.} P(A|\Omega)$ for all $A \in \mathcal{F}$.

Finally, let π be a partition of Ω—that is, a set of non-empty subsets of Ω such that every element of Ω is in exactly one such subset. Then, P is **conglomerable in** π just in case:

> For all $E \in \mathcal{F}$ and all constants k_1, k_2: if $k_1 \leq P(E|h) \leq k_2$ for all $h \in \pi$, then $k_1 \leq P(E) \leq k_2$.[9]

That is, P is conglomerable in π just in case, whenever the probability of a proposition conditional on any member of π is bounded by two values, the unconditional probability of that proposition is also bounded by those values. Say that P is **conglomerable** (simpliciter) just in case P is conglomerable in every partition of Ω; say that P is **non-conglomerable** otherwise.

A number of examples of non-conglomerable probability functions have been identified in the literature. Here I review the much-discussed example of de Finetti (1972).[10] I will formulate the example in terms of lotteries so as to facilitate later discussion.

[7] I do not assume that P is countably additive. However, I discuss the connection between countable additivity and conglomerability in Sect. 6.3.

[8] The above constraints constitute the theory of conditional probability due to de Finetti (1974) and Dubins (1975). Although the last "multiplicative" constraint is not part of Kolmogorov (1950)'s familiar axiomatization, it is a generalization of his ratio formula for conditional probability, according to which $P(A|B) = P(A \cap B)/P(B)$ when $P(B) > 0$. The multiplicative constraint applies even when $P(B) = 0$ and, as noted by Easwaran (2014), is implied by various other theories of conditional probability that allow for $P(A|B)$ to be defined when $P(B) = 0$. However, the theory of conditional probability in question—and which I will assume in what follows—differs from that developed by Kolmogorov (*ibid.*, Ch. 5). See Seidenfeld et al. (2013) for a comparison between the two theories.

[9] See de Finetti (1972, p. 99).

[10] As de Finetti (1972, pp. 98–99) notes, the example is attributed to Lévy by Cantelli (1935), though it is based on a still earlier example of de Finetti (1930). See Kadane et al. (1986) and Arntzenius et al. (2004) for additional examples.

Consider two countably infinite lotteries L_1, L_2 such that each lottery draws exactly one positive integer at random. Moreover, suppose that, for any positive integers i and j, it is possible that L_1 draws i and L_2 draws j. For every $i, j \in \mathbb{N}$, let:

- V_i = the proposition that L_1 draws i,
- H_j = the proposition that L_2 draws j, and
- $q_{i,j} = V_i \cap H_j$ = the proposition that L_1 draws i and L_2 draws j.

Note that, for every $i, j, k, l \in \mathbb{N}$, $q_{i,j} \neq \emptyset$ and $q_{i,j} \cap q_{k,l} = \emptyset$ if $i \neq k$ or $j \neq l$. Additionally, let:

- $A = [q_{1,1} \cup (q_{1,2} \cup q_{2,2}) \cup (q_{1,3} \cup q_{2,3} \cup q_{3,3}) \cup \ldots]$ = the proposition that the number drawn from L_2 is greater than or equal to that drawn from L_1,
- $\Omega = \bigcup_{i,j} q_{i,j}$ = the set of all possible outcomes, and
- $\mathcal{F}$ = the smallest Boolean algebra on Ω containing every V_i, every H_j, and A.[11]

The figure below provides a pictorial representation of $q_{i,j}, V_i, H_j$, and A.

Next, suppose that L_1 and L_2 are *fair* by the lights of some probability function P on $\mathcal{F}$. That is, suppose P satisfies the following:

> **Probabilistic Fairness.** For every $i, j \in \mathbb{N}$: $P(V_i) = P(V_j)$ and $P(H_i) = P(H_j)$.

Note that this constraint implies that $P(V_i) = P(H_j) = 0$ for every $i, j \in \mathbb{N}$.[12]

Finally, suppose that L_1 and L_2 are *independent* by the lights of P. That is, suppose P satisfies the following:

[11] That is, for any Boolean algebra $\mathcal{F}'$ on Ω containing every V_i, every H_j, and A: for all $x \in \mathcal{F}$, $x \in \mathcal{F}'$ as well.

[12] *Proof.* Suppose for reductio that $P(V_i) > 0$ for some $i \in \mathbb{N}$. Then, because P is real-valued, there is some positive integer N such that $P(V_i) > \frac{1}{N}$. Thus, by finite additivity and the assumption that L_1 is fair by the lights of P, $P(V_1 \cup \ldots \cup V_{N+1}) = (N+1)P(V_i) > 1 + \frac{1}{N}$, which exceeds 1—in violation of the axioms of probability. A similar story holds for every $P(H_j)$.

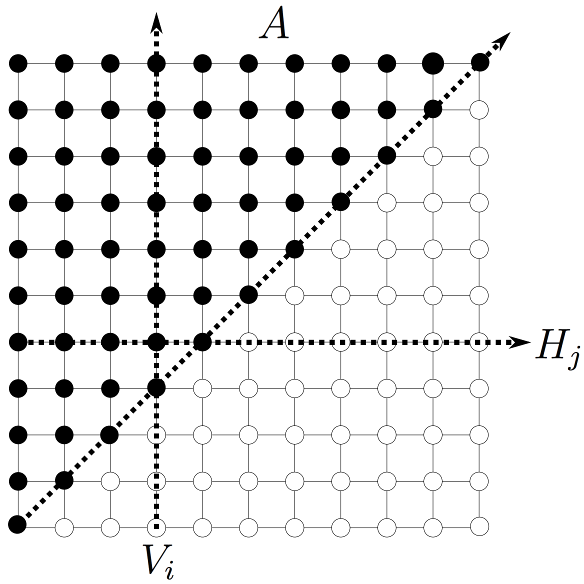

Figure 1: Each $q_{i,j}$ may be represented as a positive-integer point in the 1st quadrant of the Cartesian plane, each V_i may be represented as a "**V**ertical slice" of the 1st quadrant, each H_j may be represented as a "**H**orizontal slice" of the 1st quadrant, and A may be represented as the set of positive-integer points in the region $y \geq x$ (with the y-axis vertical and the x-axis horizontal).

> **Probabilistic Independence.** For every $i, j \in \mathbb{N}$: $P(H_j|V_i) = P(H_j)$ and $P(V_i|H_j) = P(V_i)$.[13]

Let us now ask: what is $P(A)$?

- **Answer 1.**
  Let $\pi_1 = \{V_i : i \in \mathbb{N}\}$. Note that π_1 is a partition of Ω. Now consider arbitrary $V_i \in \pi_1$. Note that $V_i = (q_{i,1} \cup q_{i,2} \cup \ldots)$ is infinite, while $V_i \cap \neg A = (q_{i,1} \cup \ldots \cup q_{i,i-1})$ is finite. (See Fig. 1 for illustration.) So,

$$P[(V_i \cap \neg A)|V_i] = P[(q_{i,1} \cup \ldots \cup q_{i,i-1})|V_i] \tag{3}$$

[13] Although the traditional "factorization" analysis takes A and B to be probabilistically independent just in case $P(A \cap B) = P(A)P(B)$, Fitelson & Hájek (2014) have forcefully argued that it is more appropriate to regard A and B as (mutually) probabilistically independent just in case $P(A|B) = P(A)$ and $P(B|A) = P(B)$. Note that, by the 4th constraint in the definition of conditional probability, these latter claims entail that $P(A \cap B) = P(A)P(B)$.

$$\begin{aligned} &= P(q_{i,1}|V_i) + \ldots + P(q_{i,i-1}|V_i) && (4)\\ &= P[(V_i \cap H_1)|V_i] + \ldots + P[(V_i \cap H_{i-1})|V_i] && (5)\\ &= P(H_1|V_i) + \ldots + P(H_{i-1}|V_i) && (6)\\ &= P(H_1) + \ldots + P(H_{i-1}) && (7)\\ &= 0 + \ldots + 0 && (8)\\ &= 0. && (9) \end{aligned}$$

The reasoning is as follows. (4) results from finite additivity. (5) uses the definition of $q_{k,l}$. (6) employs the 2nd and 4th constraints in the definition of conditional probability.[14] (7) uses **Probabilistic Independence**. (8) uses the fact that $P(H_j) = 0$ for every $j \in \mathbb{N}$. (9) is simple arithmetic. Next, by finite additivity, $P[(V_i \cap \neg A)|V_i] + P[(V_i \cap A)|V_i] = P(V_i|V_i) = 1$. Thus, $P(A|V_i) = P[(V_i \cap A)|V_i] = 1$. If P were conglomerable in π_1, then the fact that $P(A|v) = 1$ for every $v \in \pi_1$ would imply that $P(A) = 1$.

- **Answer 2.**
  Let $\pi_2 = \{H_j : j \in \mathbb{N}\}$. Note that π_2 is a partition of Ω. Now consider arbitrary $H_j \in \pi_2$. Note that $H_j = (q_{1,j} \cup q_{2,j} \cup \ldots)$ is infinite, while $H_j \cap A = (q_{1,j} \cup \ldots \cup q_{j,j})$ is finite. (See Fig. 1 for illustration.) By reasoning analogous to the above, it follows that $P[(H_j \cap A)|H_j] = 0$. Thus, $P(A|H_j) = P[(H_j \cap A)|H_j] = 0$. If P were conglomerable in π_2, then the fact that $P(A|h) = 0$ for every $h \in \pi_2$ would imply that $P(A) = 0$.

Of course, $P(A)$ cannot be both 0 and 1, so P must be non-conglomerable in at least one of π_1 and π_2. Hence:

Probabilistic Non-Conglomerability. P is non-conglomerable.

So, if a rational agent's credence function satisfies **Probabilistic Fairness** and **Probabilistic Independence**—and there seems no *prima facie* reason it couldn't—then her credence function must be non-conglomerable. However, as I said in Sect. 1, it seems *irrational* for an agent to have a non-conglomerable credence function. Hence, the (quantitative) paradox of non-conglomerability.

[14] In general, these constraints imply that $P[(A \cap B)|B] = P(A|B)P(B|B) = P(A|B)$. Thus, $P[(V_i \cap H_1)|V_i] = P(H_1|V_i), \ldots, P[(V_i \cap H_{i-1})|V_i] = P(H_{i-1}|V_i)$.

In Sect. 4, I present a qualitative analogue of this paradox.[15] Before doing so, we will need some additional theoretical machinery.

3 Comparative confidence

Comparative confidence is the attitude of being at least as confident in one proposition as in another.[16] By contrast, credence is the attitude of believing some proposition to a particular (numerical) degree. Just as a number of questions may be asked about the relation between "full" belief and credence,[17] so a number of questions may be asked about the relation between comparative confidence and credence. For example, are an agent's credences somehow reducible to, or less fundamental than, her attitudes of comparative confidence? Or are her attitudes of comparative confidence somehow reducible to, or less fundamental than, her credences? Or is neither the case?[18]

I will not assume any particular answers to such questions in what follows. Although the present paper may hold special interest for readers who subscribe to a "comparative confidence"-first view, I will merely assume that rational agents can *have* attitudes of comparative confidence. This assumption will suffice to show that there is a qualitative paradox of non-conglomerability to be reckoned with.

Notation. In what follows, I will take an agent S's comparative confidence relation $\succeq$ to be the set of S's attitudes of comparative confidence. In the unconditional case, '$A \succeq B$' means: S is at least as confident in A as she is in B. In the conditional case, '$A|B \succeq C|D$' means: S is at least as confident in A, given B, as she is in C, given D. Additionally, let '$A|B \approx C|D$' mean

[15] In Sect. 6, I discuss common responses to the quantitative paradox as they relate to possible responses to the qualitative paradox.

[16] Comparative confidence is also sometimes understood as the attitude of being strictly more confident in one proposition than another. However, taking comparative confidence to be the "at least as confident" attitude will simplify the ensuing discussion. Also, the terms 'comparative confidence', 'qualitative probability', and 'comparative probability' are often used interchangeably in the literature.

[17] See Foley (2009).

[18] Although it may no longer be so widespread, the view that comparative confidence is somehow more fundamental than credence was held by a number of notable authors in the history of probability, including Keynes (1921), de Finetti (1937), and Savage (1954). See Stefánsson (Forthcoming) for a contemporary defense of the view. See Eriksson & Hájek (2007) for a defense of the view that 'credence' is a primitive concept.

that both $A|B \succeq C|D$ and $C|D \succeq A|B$—i.e., that S is equally confident in A, given B, as in C, given D. Also, let '$A|B \succ C|D$' mean that $A|B \succeq C|D$ and it is not the case that $C|D \succeq A|B$—i.e., that S is strictly more confident in A, given B, than she is in C, given D.

A number of "representation theorems" have been proven to the effect that if an agent's comparative unconditional (conditional) confidence relation satisfies a particular set of constraints—for example, involving norms of rationality and perhaps other constraints—then it is representable by an unconditional (conditional) probability function.[19] In particular, a given comparative conditional confidence relation $\succeq$ is representable by a given conditional probability function P just in case:

Representability. $A|B \succeq C|D$ iff $P(A|B) \geq P(C|D)$.

Historically, the establishment of representation theorems has been the main motivation for studying comparative confidence. Nonetheless, I will not be concerned with the details of any representation theorem in what follows. Although I will sometimes appeal to the notion of probabilistic representability, my main concern will be with comparative confidence itself—in particular, comparative *conditional* confidence. With that said, the plan for this section is as follows.

In Sect. 3.1, I lay out some widely accepted constraints on rational comparative conditional confidence. In Sect. 3.2, I state theorems that are consequences of these constraints. In Sect. 3.3, I define a notion of conglomerability for comparative conditional confidence.

3.1 Koopman's axioms of rational comparative conditional confidence

The first axiomatization of rational comparative conditional confidence was provided by Koopman (1940a). In what follows, I will assume that any rational agent indeed satisfies Koopman's axioms. Although a number of alternative axiomatizations have since been proposed, nearly all of Koopman's

[19]See de Finetti (1937), Savage (1954), and Scott (1964) for notable representation theorems connecting comparative unconditional confidence to unconditional probability. See Koopman (1940a) and Suppes & Zanotti (1982) for notable representation theorems connecting comparative conditional confidence to conditional probability.

axioms have figured as axioms or theorems in subsequent axiomatizations.[20] As such, Koopman's axioms are widely accepted at least as rational *constraints* on comparative conditional confidence. Thus, the conclusions I will draw from Koopman's axioms are quite neutral among extant axiomatizations.

To spell out Koopman's axioms, let Ω be the set of outcomes that are epistemically possible for agent S.[21] Also, let $\mathcal{F}$ be a Boolean algebra on Ω—intuitively, the set of propositions that are entertainable by S—and let $\mathcal{F}_0$ be the set of non-empty elements of $\mathcal{F}$. Then, S's comparative conditional confidence relation $\succeq$ is a binary relation on $\mathcal{F} \times \mathcal{F}_0$ that satisfies the following axioms. For simplicity, I leave quantification over propositions implicit in the axioms and theorems that follow.

1. **Verified Contingency.** $k|k \succeq a|h$.
2. **Implication.** If $a|h \succeq k|k$, then $h \subseteq a$.
3. **Reflexivity.** $a|h \succeq a|h$.
4. **Transitivity.** If $c|l \succeq b|k$ and $b|k \succeq a|h$, then $c|l \succeq a|h$.
5. **Antisymmetry.** If $b|k \succeq a|h$, then $\neg a|h \succeq \neg b|k$.
6. **Composition.**
   Suppose:
   (a) $(a_1 \cap b_1) \cap h_1 \neq \emptyset$ and $(a_2 \cap b_2) \cap h_2 \neq \emptyset$.
   (b) $a_2|h_2 \succeq a_1|h_1$.
   (c) $b_2|(a_2 \cap h_2) \succeq b_1|(a_1 \cap h_1)$.

   Then: $(a_2 \cap b_2)|h_2 \succeq (a_1 \cap b_1)|h_1$.[22]

[20] See Krantz et al. (1971, pp. 221-222) for general discussion of similarities among Koopman's axiomatization and other axiomatizations. More specifically, see Luce (1968, pp. 483-484) for discussion of logical connections among his own axioms and Koopman's axioms. Additionally, the axiomatization of Hawthorne (Forthcoming) is nearly identical to that of Koopman.

[21] Although Koopman does not speak in terms of 'epistemic possibilities', this terminology will be useful in what follows.

[22] Koopman actually states two axioms of composition; the other is analogous.

7. **Decomposition.**
   Suppose:

   (a) $(a_1 \cap b_1) \cap h_1 \neq \emptyset$ and $(a_2 \cap b_2) \cap h_2 \neq \emptyset$.

   (b) $a_1|h_1 \succeq a_2|h_2$.

   (c) $(a_2 \cap b_2)|h_2 \succeq (a_1 \cap b_1)|h_1$.

   Then: $b_2|(a_2 \cap h_2) \succeq b_1|(a_1 \cap h_1)$.[23]

8. **Alternative Presumption.** If $r|s \succeq a|(b \cap h)$ and $r|s \succeq a|(\neg b \cap h)$, then $r|s \succeq a|h$.[24]

9. **Subdivision.**
   Suppose the propositions $a_1, \ldots, a_n, b_1, \ldots, b_n$ are such that:

   (a) $a_i \cap a_j = b_i \cap b_j = \emptyset$ for all $i, j = 1, \ldots, n$ such that $i \neq j$.

   (b) $a = (a_1 \cup \ldots \cup a_n) \neq \emptyset$ and $b = (b_1 \cup \ldots \cup b_n) \neq \emptyset$.

   (c) $a_n|a \succeq \ldots \succeq a_1|a$ and $b_n \succeq \ldots \succeq b_1|b$.

   Then: $b_n|b \succeq a_1|a$.[25]

3.2 Theorems of rational comparative conditional confidence

In this section, I state three theorems (or straightforward consequences of theorems) proven in Koopman (1940a) on the basis of the above axioms. I will appeal to them in Sect. 4.

The first two theorems are relatively straightforward.

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[23]Koopman actually states four axioms of decomposition; the others are analogous.

[24]This axiom ensures that any rational comparative conditional confidence relation is conglomerable in any finite partition (in the sense I describe in Sect. 3.3). In Sect. 4, I describe a comparative conditional confidence relation that satisfies Koopman's axioms but is non-conglomerable in some *infinite* partition. As it turns out, none of my argument will appeal to this axiom. However, I include it here for completeness.

[25]As Koopman (1940b) notes, both **Alternative Presumption** and **Subdivision** are entailed by the other axioms along with the assumption that $\succeq$ is complete—that is, that $p|q \succeq r|s$ or $r|s \succeq p|q$. However, I will not assume that $\succeq$ is complete in what follows.

Theorem 1. $(p \cap q)|q \approx p|q$.[26]

Theorem 2. *If* $p \subseteq q$ *and* $q \subseteq r$*, then* $p|q \succeq p|r$.[27]

To spell out the third theorem, we will need a definition and an assumption.

Definition. For any positive integer n, let an n**-scale** be a set of n propositions $\{u_1, \ldots, u_n\}$ such that:

1. $u = (u_1 \cup \ldots \cup u_n) \neq \emptyset$.
2. $u_i \cap u_j = \emptyset$ for all $i, j = 1, \ldots, n$ such that $i \neq j$.
3. $u_i|u \approx u_j|u$ for all $i, j = 1, \ldots, n$.[28]

That is, an n-scale for agent S is a non-empty set of n mutually disjoint propositions such that S is equally confident in each of them given the union of all of them. Intuitively, if Ω is the set of epistemically possible outcomes for S, then an n-scale is a finite sub-lottery of Ω that S treats as *qualitatively* fair.

Assumption. For every positive integer n, there is at least one n-scale.[29]

This assumption, unlike the axioms of Sect. 3.1, should not be viewed as a rational constraint on comparative conditional confidence. Rather, it is a contingent claim about an agent's epistemic context. In Sect. 4, I will describe an epistemic context in which it holds.

The next theorem, which follows from Koopman's axioms and **Assumption**, illuminates a connection between n-scales and fractions.

Theorem 3. *Suppose:*

1. $\{u_1, \ldots, u_n\}$ *is an* n*-scale and* $\{v_1, \ldots, v_m\}$ *is an* m*-scale.*

[26]This is a straightforward consequence of Koopman's Theorem 3, the first part of which states that if $(a \cap h) \subseteq (b \cap h)$, then $b|h \succeq a|h$. His proof employs **Verified Contingency**, **Implication**, **Reflexivity**, **Antisymmetry**, and **Composition**. To establish that $(p \cap q)|q \succeq p|q$, it suffices to set $a = p, b = p \cap q$, and $h = q$ in the theorem. To establish that $p|q \succeq (p \cap q)|q$, it suffices to set $a = p \cap q, b = p$, and $h = q$.

[27]This is the first part of Koopman's Theorem 4. His proof employs **Verified Contingency**, **Implication**, **Reflexivity**, **Antisymmetry**, and **Composition**.

[28]This is Koopman's Definition 1.

[29]This is Koopman's Assumption (stated separately from his axioms).

2. *ρ and σ are integers such that $0 \leq \sigma \leq n$ and $0 \leq \rho \leq m$.*

3. $\frac{\sigma}{n} \geq \frac{\rho}{m}$.

Then:

$$(u_{i_1} \cup \ldots \cup u_{i_\sigma})|u \succeq (v_{j_1} \cup \ldots \cup v_{j_\rho})|v. \tag{10}$$

If $\frac{\sigma}{n} > \frac{\rho}{m}$, then replace '$\succeq$' with '$\succ$' in the above.[30]

In what follows, I will write $P^*[(u_{i_1} \cup \ldots \cup u_{i_\sigma})|u] = \frac{\sigma}{n}$ and $P^*[(v_{j_1} \cup \ldots \cup v_{j_\rho})|v] = \frac{\rho}{m}$ when the above conditions hold. For example, if $\{u_1, u_2\}$ is a 2-scale and $\{v_1, v_2, v_3\}$ is a 3-scale, then $P^*[u_1|(u_1 \cup u_2)] = \frac{1}{2}$ and $P^*[(v_1 \cup v_2)|(v_1 \cup v_2 \cup v_3)] = \frac{2}{3}$. Thus, n-scales behave intuitively like fractional probabilities.

3.3 Qualitative conglomerability

By analogy with the definition of probabilistic conglomerability, we may define conglomerability for an agent S's comparative conditional confidence relation $\succeq$ as follows. Let Ω be the set of outcomes that are epistemically possible for S, $\mathcal{F}$ a Boolean algebra on Ω, $\mathcal{F}_0$ the set of non-empty elements of $\mathcal{F}$, and $\succeq$ a binary relation on $\mathcal{F} \times \mathcal{F}_0$. Also, let π be a partition of Ω. Then, $\succeq$ is **conglomerable in** π just in case:

> For all $E, p_1, p_2 \in \mathcal{F}$ and $q_1, q_2 \in \mathcal{F}_0$: if $p_2|q_2 \succeq E|h \succeq p_1|q_1$ for all $h \in \pi$, then $p_2|q_2 \succeq E|\Omega \succeq p_1|q_1$.

Note that the quantitative comparisons of probability which figured in the definition of probabilistic conglomerability have now been replaced with qualitative comparisons of conditional confidence.

Say that $\succeq$ is **conglomerable** (simpliciter) just in case $\succeq$ is conglomerable in every partition of Ω; say that $\succeq$ is **non-conglomerable** otherwise.

4 Qualitative non-conglomerability

In this section, I present a qualitative analogue of the (quantitative) paradox of non-conglomerability. In Sects. 4.1–4.2, I show that de Finetti's non-conglomerable probability function from Sect. 2 can be reformulated as a

[30] This is proven as Koopman's Theorem 14. His proof employs **Verified Contingency**, **Antisymmetry**, **Composition**, **Decomposition**, **Subdivision**, and **Assumption**.

comparative confidence relation that satisfies Koopman's axioms yet is non-conglomerable. In Sect. 4.3, I discuss the paradoxical nature of this result.

The setup is analogous to that of Sect. 2. First, let S be some agent who satisfies Koopman's axioms. As before, let L_1 and L_2 be countably infinite lotteries such that, for any positive integers i and j, it is epistemically possible for S that L_1 draws i and L_2 draws j. Also, let:

- V_i = the proposition that L_1 draws i,
- H_j = the proposition that L_2 draws j,
- $q_{i,j} = V_i \cap H_j$ = the proposition that L_1 draws i and L_2 draws j,
- $A = [q_{1,1} \cup (q_{1,2} \cup q_{2,2}) \cup (q_{1,3} \cup q_{2,3} \cup q_{3,3}) \cup \ldots]$ = the proposition that the number drawn from L_2 is greater than or equal to that drawn from L_1,
- $\Omega = \bigcup_{i,j} q_{i,j}$ = the set of all epistemically possible outcomes for S, and
- $\mathcal{F}$ = the smallest Boolean algebra on Ω containing every V_i, every H_j, and A.

Let $\succeq$ be S's comparative conditional confidence relation on $\mathcal{F} \times \mathcal{F}_0$.

Next, suppose that each lottery is *qualitatively* fair in the sense that in the sense that S is just as confident in any integer being drawn by L_1 as any other, and S is just as confident in any integer being drawn by L_2 as any other. That is:

> **Fairness.** For every $i, j \in \mathbb{N}$: $V_i|\Omega \approx V_j|\Omega$ and $H_i|\Omega \approx H_j|\Omega$.

Finally, suppose that L_1 and L_2 are *qualitatively* independent of one another. That is:

> **Independence.** For every $i, j \in \mathbb{N}$: $V_i|H_j \approx V_i|\Omega$ and $H_i|V_j \approx H_i|\Omega$.[31]

I now show that **Fairness** and **Independence**, in conjunction with Koopman's axioms, entail that $\succeq$ is non-conglomerable. I begin with four lemmas.

[31] **Independence** employs the notion of independence for comparative conditional confidence described by Krantz et al. (1971, p. 238).

4.1 Preliminary lemmas

Lemma 1 states that, unconditionally, S should be equally confident in every possible outcome in Ω. Hence, S may regard $\Omega = \{q_{1,1}, q_{1,2}, \ldots\}$ as the set of possible outcomes of an *individual* lottery that is countably infinite and qualitatively fair.

Lemma 1. For every $i, j, k, l \in \mathbb{N} : q_{i,j}|\Omega \approx q_{k,l}|\Omega$.

Proof. Let i, j, k, l be arbitrary positive integers. By **Fairness** and **Independence**, it follows that $H_j|V_i \approx H_j|\Omega \approx H_l|\Omega \approx H_l|V_k$. By **Transitivity**, $H_j|V_i \approx H_l|V_k$.

Now set $a_1 = V_k, b_1 = H_l, h_1 = \Omega, a_2 = V_i, b_2 = H_j$, and $h_2 = \Omega$. Then, $(a_1 \cap b_1) \cap h_1 = (V_k \cap H_l) \cap \Omega = q_{k,l} \neq \emptyset$ and $(a_2 \cap b_2) \cap h_2 = (V_i \cap H_j) \cap \Omega = q_{i,j} \neq \emptyset$. Next, by **Fairness**, $V_i|\Omega \approx V_k|\Omega$. So, $a_2|h_2 \succeq a_1|h_1$. Further, since $V_i = V_i \cap \Omega$ and $V_k = V_k \cap \Omega$, the fact that $H_j|V_i \approx H_l|V_k$ entails that $H_j|(V_i \cap \Omega) \approx H_l|(V_k \cap \Omega)$. So, $b_2|(a_2 \cap h_2) \succeq b_1|(a_1 \cap h_1)$. Thus, by **Composition**, $(V_i \cap H_j)|\Omega \succeq (V_k \cap H_l)|\Omega$. Similarly, setting $a_1 = V_i, b_1 = H_j, h_1 = \Omega, a_2 = V_k, b_2 = H_l$, and $h_2 = \Omega$, **Composition** yields that $(V_k \cap H_l)|\Omega \succeq (V_i \cap H_j)|\Omega$. Thus, $(V_i \cap H_j)|\Omega \approx (V_k \cap H_l)|\Omega$. That is, $q_{i,j}|\Omega \approx q_{k,l}|\Omega$. □ □

Lemma 2 states that, for any positive integer n, any n-member subset of Ω is an n-scale for S. Hence, S should regard each finite *sub*-lottery of Ω as qualitatively fair as well. Lemma 2 entails that **Assumption** from Sect. 3.2 holds in the epistemic context in question. Thus, Theorem 3 holds as well.

Lemma 2. Let $B = \{q_{i_1,j_1}, \ldots, q_{i_n,j_n}\}$, for arbitrary positive integers $i_1, \ldots, i_n$, $j_1, \ldots, j_n$. Then, B is an n-scale.

Proof. Let $u_k = q_{i_k,j_k}$ for all $k = 1, \ldots, n$, and let $u = (u_1 \cup \ldots \cup u_n)$. Since $q_{i,j} \neq \emptyset$ for every $i, j \in \mathbb{N}$, $u \neq \emptyset$. Also, since $q_{i,j} \cap q_{k,l} = \emptyset$ for all $i, j, k, l \in \mathbb{N}$ such that $i \neq k$ or $j \neq l$, $u_i \cap u_j = \emptyset$ for all $i, j = 1, \ldots, n$ such that $i \neq j$.

Now set $a_1 = u, b_1 = u_l, h_1 = \Omega, a_2 = u, b_2 = u_k$, and $h_2 = \Omega$ for arbitrary $k, l \leq n$. Note that $(a_1 \cap b_1) \cap h_1 = (u \cap u_l) \cap \Omega = u_l \cap \Omega = q_{i_l,j_l} \cap \Omega = q_{i_l,j_l} \neq \emptyset$. Similarly, $(a_2 \cap b_2) \cap h_2 = (u \cap u_k) \cap \Omega = u_k \cap \Omega = q_{i_k,j_k} \neq \emptyset$. Also, by **Reflexivity**, $u|\Omega \succeq u|\Omega$. So, $a_1|h_1 \succeq a_2|h_2$. Next, since $u \cap u_k = q_{i_k,j_k}$ and $u \cap u_l = q_{i_l,j_l}$, Lemma 1 entails that $(u \cap u_k)|\Omega \approx (u \cap u_l)|\Omega$. So, $(a_2 \cap b_2)|h_2 \succeq (a_1 \cap b_1)|h_1$. Thus, by **Decomposition**, $u_k|(u \cap \Omega) \succeq u_l|(u \cap \Omega)$. Since $u \cap \Omega =$

$(u_1 \cup \ldots \cup u_n) \cap \Omega = (q_{i_1,j_1} \cup \ldots \cup q_{i_n,j_n}) \cap \Omega = (q_{i_1,j_1} \cup \ldots \cup q_{i_n,j_n}) = u$, it follows that $u_k|u \succeq u_l|u$. Similarly, reversing b_1 and b_2, **Decomposition** yields that $u_l|u \succeq u_k|u$. As a result, $u_k|u \approx u_l|u$. That is, $q_{i_k,j_k}|(q_{i_1,j_1} \cup \ldots \cup q_{i_n,j_n}) \approx q_{i_l,j_l}|(q_{i_1,j_1} \cup \ldots \cup q_{i_n,j_n})$. Thus, $B = \{q_{i_1,j_1}, \ldots, q_{i_n,j_n}\}$ is an n-scale. □ □

Note that, using the notation of Sect. 3.2, Lemma 2 entails that $P^*(q_{1,1}|q_{1,1})$ $= 1$ and $P^*[q_{1,1}|(q_{1,1} \cup q_{1,2})] = \frac{1}{2}$. Intuitively, Lemma 3 corresponds to the claim that S's credence in A, given any V_i, should lie between $\frac{1}{2}$ and 1.

Lemma 3. For every $i \in \mathbb{N}$: $q_{1,1}|q_{1,1} \succeq A|V_i \succeq q_{1,1}|(q_{1,1} \cup q_{1,2})$.

Proof. Note that $q_{1,1}|q_{1,1} \succeq A|V_i$ by **Verified Contingency**. To show that $A|V_i \succeq q_{1,1}|(q_{1,1} \cup q_{1,2})$, it suffices (by **Antisymmetry**) to show that $\neg q_{1,1}|(q_{1,1} \cup q_{1,2}) \succeq \neg A|V_i$.

First, let i be an arbitrary positive integer such that $i > 1$, $Q_{1,2} = (q_{1,1} \cup q_{1,2})$, and $Q_{i,2(i-1)} = (q_{i,1} \cup \ldots \cup q_{i,2(i-1)})$. By Lemma 2, $\{q_{1,1}, q_{1,2}\}$ is a 2-scale and $\{q_{i,1}, \ldots, q_{i,2(i-1)}\}$ is a $2(i-1)$-scale. Now set $n = 2, \sigma = 1, m = 2(i-1)$, and $\rho = (i-1)$. Since $\frac{\sigma}{n} \geq \frac{\rho}{m}$, Theorem 3 yields that

$$q_{1,2}|Q_{1,2} \succeq (q_{i,1} \cup \ldots \cup q_{i,i-1})|Q_{i,2(i-1)}. \tag{11}$$

Next, note that $V_i = (q_{i,1} \cup q_{i,2} \cup \ldots) = (Q_{i,2(i-1)} \cup q_{i,2(i-1)+1} \cup \ldots)$. So, $(q_{i,1} \cup \ldots \cup q_{i,i-1}) \subseteq Q_{i,2(i-1)}$ and $Q_{i,2(i-1)} \subseteq V_i$. Hence, by Theorem 2,

$$(q_{i,1} \cup \ldots \cup q_{i,i-1})|Q_{i,2(i-1)} \succeq (q_{i,1} \cup \ldots \cup q_{i,i-1})|V_i. \tag{12}$$

Recall (cf. Sect. 2) that $V_i \cap \neg A = (q_{i,1} \cup \ldots \cup q_{i,i-1})$. So,

$$\begin{aligned}(q_{i,1} \cup \ldots \cup q_{i,i-1})|Q_{i,2(i-1)} &\succeq (V_i \cap \neg A)|V_i &(13)\\ &\succeq \neg A|V_i, &(14)\end{aligned}$$

using Theorem 1.

Next, note that $q_{1,2} = (\neg q_{1,1} \cap Q_{1,2})$. By Theorem 1, $(\neg q_{1,1} \cap Q_{1,2})|Q_{1,2} \approx \neg q_{1,1}|Q_{1,2}$. As a result, $q_{1,2}|Q_{1,2} \approx \neg q_{1,1}|Q_{1,2}$. So, $\neg q_{1,1}|Q_{1,2} \succeq q_{1,2}|Q_{1,2}$. Putting everything together:

$$\begin{aligned}\neg q_{1,1}|Q_{1,2} &\succeq q_{1,2}|Q_{1,2} &(15)\\ &\succeq (q_{i,1} \cup \ldots \cup q_{i,i-1})|Q_{i,2(i-1)} &(16)\\ &\succeq (V_i \cap \neg A)|V_i &(17)\\ &\succeq \neg A|V_i. &(18)\end{aligned}$$

Thus, by **Transitivity**, $\neg q_{1,1}|Q_{1,2} \succeq \neg A|V_i$. Finally, by **Antisymmetry**, $A|V_i \succeq q_{1,1}|Q_{1,2}$. That is, $A|V_i \succeq q_{1,1}|(q_{1,1} \cup q_{1,2})$. □ □

Note that, by Lemma 2, $P^*[q_{1,1}|(q_{1,1} \cup q_{1,2} \cup q_{1,3})] = \frac{1}{3}$. Intuitively, Lemma 4 corresponds to the claim that S's credence in A, given any H_i, should lie between 0 and $\frac{1}{3}$.

Lemma 4. For every $i \in \mathbb{N}$: $q_{1,1}|(q_{1,1} \cup q_{1,2} \cup q_{1,3}) \succeq A|H_i \succeq \neg q_{1,1}|q_{1,1}$.

Proof. The proof is analogous to that of Lemma 3. First, let i be an arbitrary positive integer, $Q_{1,3} = (q_{1,1} \cup q_{1,2} \cup q_{1,3})$, and $Q_{3i,i} = (q_{1,i} \cup \ldots \cup q_{3i,i})$. By Lemma 2, $\{q_{1,1}, q_{1,2}, q_{1,3}\}$ is a 3-scale and $\{q_{1,i}, \ldots, q_{3i,i}\}$ is a $3i$-scale. Now set $n = 3, \sigma = 1, m = 3i$, and $\rho = i$. Since $\frac{\sigma}{n} \geq \frac{\rho}{m}$, a similar application of Theorem 3 yields that

$$q_{1,1}|Q_{1,3} \succeq (q_{1,i} \cup \ldots \cup q_{i,i})|Q_{3i,i}. \tag{19}$$

Next, note that $H_i = (q_{1,i} \cup q_{2,i} \cup \ldots) = (Q_{3i,i} \cup q_{3i+1,i} \cup \ldots)$. So, $(q_{1,i} \cup \ldots \cup q_{i,i}) \subseteq Q_{3i,i}$ and $Q_{3i,i} \subseteq H_i$. Hence, by Theorem 2,

$$(q_{1,i} \cup \ldots \cup q_{i,i})|Q_{3i,i} \succeq (q_{1,i} \cup \ldots \cup q_{i,i})|H_i. \tag{20}$$

Recall (cf. Sect. 2) that $H_i \cap A = (q_{1,i} \cup \ldots \cup q_{i,i})$. So,

$$\begin{aligned} (q_{1,i} \cup \ldots \cup q_{i,i})|Q_{3i,i} &\succeq (H_i \cap A)|H_i & (21)\\ &\succeq A|H_i, & (22) \end{aligned}$$

using Theorem 1. Putting everything together:

$$\begin{aligned} q_{1,1}|Q_{1,3} &\succeq (q_{1,i} \cup \ldots \cup q_{i,i})|Q_{3i,i} & (23)\\ &\succeq (H_i \cap A)|H_i & (24)\\ &\succeq A|H_i. & (25) \end{aligned}$$

Thus, by **Transitivity**, $q_{1,1}|Q_{1,3} \succeq A|H_i$. That is, $q_{1,1}|(q_{1,1} \cup q_{1,2} \cup q_{1,3}) \succeq A|H_i$.

Finally, by **Verified Contingency**, $q_{1,1}|q_{1,1} \succeq \neg A|H_i$. By **Antisymmetry**, $\neg\neg A|H_i \succeq \neg q_{1,1}|q_{1,1}$. Since $\neg\neg A = A$, $A|H_i \succeq \neg q_{1,1}|q_{1,1}$. □ □

4.2 Main technical result

I now prove the main technical result of the paper:

Qualitative Non-Conglomerability. $\succeq$ *is non-conglomerable.*

Proof. As in Sect. 2, let $\pi_1 = \{V_i : i \in \mathbb{N}\}$ and $\pi_2 = \{H_j : j \in \mathbb{N}\}$. Then, by Lemmas 3 and 4:

- $q_{1,1}|q_{1,1} \succeq A|v \succeq q_{1,1}|(q_{1,1} \cup q_{1,2})$ for all $v \in \pi_1$, and
- $q_{1,1}|(q_{1,1} \cup q_{1,2} \cup q_{1,3}) \succeq A|h \succeq \neg q_{1,1}|q_{1,1}$ for all $h \in \pi_2$.

Next, set $n = 2, \sigma = 1, m = 3$, and $\rho = 1$. Since $\frac{\sigma}{n} > \frac{\rho}{m}$, Theorem 3 yields that

$$q_{1,1}|(q_{1,1} \cup q_{1,2}) \succ q_{1,1}|(q_{1,1} \cup q_{1,2} \cup q_{1,3}). \tag{26}$$

Now suppose for reductio that $\succeq$ is conglomerable in both π_1 and π_2. Then, $q_{1,1}|q_{1,1} \succeq A|\Omega \succeq q_{1,1}|(q_{1,1} \cup q_{1,2})$ and $q_{1,1}|(q_{1,1} \cup q_{1,2} \cup q_{1,3}) \succeq A|\Omega \succeq \neg q_{1,1}|q_{1,1}$. As a result,

$$\begin{aligned} A|\Omega &\succeq q_{1,1}|(q_{1,1} \cup q_{1,2}) && (27)\\ &\succ q_{1,1}|(q_{1,1} \cup q_{1,2} \cup q_{1,3}). && (28) \end{aligned}$$

So, by **Transitivity** (and the definition of '$\succ$'), $A|\Omega \succ q_{1,1}|(q_{1,1} \cup q_{1,2} \cup q_{1,3})$. But we just saw that $q_{1,1}|(q_{1,1} \cup q_{1,2} \cup q_{1,3}) \succeq A|\Omega$. Contradiction. It follows that $\succeq$ is non-conglomerable in at least one of π_1 and π_2. Hence, $\succeq$ is non-conglomerable. □ □

Remark. Although **Qualitative Non-Conglomerability** is a consequence of Koopman's axioms, **Fairness**, and **Independence**, it should be noted that **Independence** was not essential to the proof of this result. Because the possible outcomes of any countably infinite lottery can be labeled with the pairs of positive integers (since both sets are countably infinite), any comparative confidence relation that treats some countably infinite lottery as fair satisfies Lemma 1 with respect to some labeling of this sort. Since **Independence** was only used in proving Lemma 1, it was therefore not essential to the above proof. Hence, non-conglomerability afflicts any agent who satisfies Koopman's axioms and treats some countably infinite lottery as qualitatively fair.

4.3 The qualitative paradox of non-conglomerability

Although Koopman's axioms of rational comparative confidence do not enjoy the popularity of Kolmogorov's axioms of probability, they seem quite plausible upon reflection. Additionally, there seems no *prima facie* reason a rational agent couldn't treat some countably infinite lottery as qualitatively fair. As I will explain, however, the consequence that a rational agent can have a non-conglomerable comparative confidence relation is quite counterintuitive. This consequence is the *qualitative* paradox of non-conglomerability.

Consider the following inference rule of deductive logic:

> **Proof by Cases.**
> Suppose:
>
> 1. $Q_1 \to R$.
> 2. $Q_2 \to R$.
>
> Then: $(Q_1 \vee Q_2) \to R$.[32]

Proof by Cases is as intuitively plausible an inference rule as any. The following constraint on rational comparative conditional confidence is an epistemic cousin of it:

> **Confidence by Cases.**
> Suppose:
>
> 1. $a|b_1 \succeq r|s$.
> 2. $a|b_2 \succeq r|s$.
>
> Then: $a|(b_1 \cup b_2) \succeq r|s$.[33]

Let S be an arbitrary rational agent. In words, **Confidence by Cases** says:

> **Confidence by Cases.**
> Suppose:

[32] Strictly speaking, this inference rule results from applying the deduction theorem to what is ordinarily called "proof by cases". However, this inference rule will be more relevant than the latter in what follows.

[33] **Confidence by Cases** is a special case of Koopman's axiom of **Alternative Presumption.** However, I have renamed the latter for illustrative purposes.

1. S is at least as confident in a, given b_1, as she is in r, given s.
2. S is at least as confident in a, given b_2, as she is in r, given s.

Then: S is at least as confident in a, given b_1 or b_2, as she is in r, given s.

Although **Confidence by Cases** may not seem as intuitively compelling as **Proof by Cases**—if only because it involves the relatively unfamiliar notion of comparative conditional confidence—it still seems quite plausible.

Next, consider the following infinitary version of **Proof by Cases**:

Infinitary Proof by Cases.
Suppose:

1. $Q_1 \to R$.
2. $Q_2 \to R$.

   $\vdots$

Then: $(Q_1 \vee Q_2 \vee \ldots) \to R$.

Infinitary Proof by Cases seems no less plausible than its finite version (provided that we permit infinitary inferences). It is structurally similar to the following infinitary version of **Confidence by Cases**:

Infinitary Confidence by Cases.
Suppose:

1. $a|b_1 \succeq r|s$.
2. $a|b_2 \succeq r|s$.

   $\vdots$

Then: $a|(b_1 \cup b_2 \cup \ldots) \succeq r|s$.

Infinitary Confidence by Cases seems no less intuitively plausible than its finite version (provided that we permit rational agents to have infinitely many attitudes, if only dispositionally). Nonetheless, it conflicts with **Qualitative Non-Conglomerability**.

To see this, let $r = q_{1,1}$, $s = (q_{1,1} \cup q_{1,2})$, and $t = (q_{1,1} \cup q_{1,2} \cup q_{1,3})$. Note that Lemma 3 entails the following:

1. $A|V_1 \succeq r|s$.
2. $A|V_2 \succeq r|s$.

   $\vdots$

So, by **Infinitary Confidence by Cases**, we would intuitively expect:

$$A|(V_1 \cup V_2 \cup \ldots) \succeq r|s. \tag{29}$$

Similarly, Lemma 4 entails:

1. $r|t \succeq A|H_1$.
2. $r|t \succeq A|H_2$.

   $\vdots$

So, by **Infinitary Confidence by Cases**, we would intuitively expect:

$$r|t \succeq A|(H_1 \cup H_2 \cup \ldots). \tag{30}$$

However, **Qualitative Non-Conglomerability** entails that at least one of these intuitive expectations is false.[34] Hence, **Infinitary Confidence by Cases** must fail as well.

Although this consequence is not a violation of deductive logic, it is still surprising. Just as the infinitary version of **Proof by Cases** seems intuitively plausible, so does the infinitary version of **Confidence by Cases**. It seems quite odd that the infinitary version of one should be true while the infinitary version of the other is false. Yet this is a consequence of Koopman's intuitively plausible axioms and the intuitively plausible assumption that a rational agent can treat a countably infinite lottery as qualitatively fair. Hence, the qualitative paradox of non-conglomerability.[35]

[34] *Proof.* We saw that $\succeq$ is non-conglomerable in at least one of π_1 and π_2. Suppose that $\succeq$ is non-conglomerable in π_1. Then, it is not the case that $q_{1,1}|q_{1,1} \succeq A|\Omega \succeq q_{1,1}|(q_{1,1} \cup q_{1,2})$. However, by **Reflexivity**, $q_{1,1}|q_{1,1} \succeq A|\Omega$. Hence, $A|\Omega \not\succeq q_{1,1}|(q_{1,1} \cup q_{1,2})$. That is, $A|\Omega \not\succeq r|s$. Since $\Omega = (V_1 \cup V_2 \cup \ldots)$, it follows that $A|(V_1 \cup V_2 \cup \ldots) \not\succeq r|s$. Next, suppose that $\succeq$ is non-conglomerable in π_2. Then, it is not the case that $q_{1,1}|(q_{1,1} \cup q_{1,2} \cup q_{1,3}) \succeq A|\Omega \succeq \neg q_{1,1}|q_{1,1}$. By **Verified Contingency**, $q_{1,1}|q_{1,1} \succeq \neg A|\Omega$. So, by **Antisymmetry**, $A|\Omega \succeq \neg q_{1,1}|q_{1,1}$. Hence, it is not the case that $q_{1,1}|(q_{1,1} \cup q_{1,2} \cup q_{1,3}) \succeq A|\Omega$. That is, $r|t \not\succeq A|\Omega$. Since $\Omega = (H_1 \cup H_2 \cup \ldots)$, it follows that $r|t \not\succeq A|(H_1 \cup H_2 \cup \ldots)$.

[35] Although de Finetti (1972, p. 104) only considered probabilistic non-conglomerability

5 Philosophical significance of the qualitative paradox

In this section, I argue that the qualitative paradox of non-conglomerability has distinctive philosophical significance for at least three reasons.

First, the qualitative paradox exposes a new kind of non-conglomerability. Second, the qualitative paradox entails the quantitative paradox, although the converse is not the case. Third, the qualitative paradox has relevance to infinitesimals and the characterization of probabilistically fair, infinite lotteries.

5.1 Reason 1: Qualitative paradox exposes a new kind of non-conglomerability

Because the quantitative paradox of Sect. 2 involves credence and the qualitative paradox of Sect. 4 involves comparative confidence, the latter involves a distinct kind of non-conglomerability. As past research has focused exclusively on probabilistic non-conglomerability, the qualitative paradox exposes a new (albeit analogous) avenue of potential research—namely, investigation into the sources and varieties of qualitative non-conglomerability. I discuss some open questions in Sect. 7.

It may be unsurprising that a kind of non-conglomerability can arise in a qualitative setting. In particular, it might be thought that the mere existence of de Finetti's non-conglomerable probability function P entails the existence of a rational comparative confidence relation—for example, one that is representable by P—that is non-conglomerable in an analogous manner. As such, it might be thought that there are no distinctive questions

to be in superficial—but not actual—violation of deductive logic as well, it should be remembered that his view of probability was deeply operationalist. Famously, de Finetti held that all statements of probability were to be understood ultimately in terms of betting ratios. Hence, from an operationalist perspective, probabilistic non-conglomerability may not appear so odd because its consequences for betting behavior do not so saliently violate analogues of deductive logic. By contrast, qualitative non-conglomerability involves real doxastic attitudes that violate a clear epistemic analogue of a deductive inference rule. Thus, qualitative non-conglomerability is apt to appear more odd than de Finetti thought probabilistic non-conglomerability to be. Probabilistic non-conglomerability may similarly appear more odd under less operationalist, more realist interpretations of subjective probability.

of non-conglomerability to be pursued at the qualitative level because they can all be pursued in an analogous fashion at the quantitative level.

As I will argue, however, these thoughts are mistaken. First, there is no rational comparative confidence relation that is representable by P and is analogously non-conglomerable because P cannot represent *any* rational comparative confidence relation.[36] Second, there is not even any rational comparative confidence relation that is "almost" representable by P (in the sense I describe below) and is analogously non-conglomerable. Thus, the quantitative non-conglomerability result of Sect. 2 does not, in any straightforward sense, entail the qualitative paradox.

Informally, my argument turns on the fact that P makes no discriminations among propositions of probability 0 (or 1), whereas all rational comparative confidence relations do make at least some discriminations among such propositions. Since the manner in which we saw P to be non-conglomerable entails treating all propositions of probability 0 (or 1) as on par, it will follow that there is no rational comparative confidence relation that is non-conglomerable in a manner entirely analogous to that in which P is non-conglomerable. I now state matters more precisely.

First, suppose for reductio that P can indeed represent some rational comparative confidence relation $\succeq$. As we saw in Sect. 2, $P(V_1|\Omega) = P(V_2|\Omega) = 0$. So, by finite additivity, $P[(V_1 \cup V_2)|\Omega] = 0$. Next, by **Representability**, $V_1|\Omega \approx V_2|\Omega \approx (V_1 \cup V_2)|\Omega$. However, it is a consequence of Koopman's axioms that $(V_1 \cup V_2)|\Omega \succ V_1|\Omega$.[37] Contradiction. Thus, P cannot represent any rational comparative confidence relation.

Next, suppose there is some rational comparative confidence $\succeq$ that is "almost" representable by P. That is, suppose the following holds:

> **Almost Representability.** If $A|B \succeq C|D$, then $P(A|B) \geq P(C|D)$.

In the Appendix, I show that any (conditional) probability function can almost represent some rational comparative confidence relation and, in fact, P

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[36] Here, and throughout Sect. 5, I use the term 'rational comparative confidence relation' to denote any comparative confidence relation that satisfies Koopman's axioms. In Sect. 6, I discuss the question of whether the non-conglomerable comparative confidence relation of Sect. 4 is indeed rationally permissible.

[37] This claim is a straightforward consequence of Theorem 3 in Koopman (1940a), the second part of which states that if $(a \cap h)$ is a proper subset of $(b \cap h)$, then $b|h \succ a|h$. Note that $(V_1 \cap \Omega)$ is a proper subset of $(V_1 \cup V_2) \cap \Omega$. Thus, setting $a = V_1, b = (V_1 \cup V_2)$, and $h = \Omega$ yields that $(V_1 \cup V_2)|\Omega \succ V_1|\Omega$.

can almost represent the comparative confidence relation of Sect. 4. Nonetheless, as I will now show, there is no rational comparative confidence relation that is almost representable by P and is analogously non-conglomerable.

Let i, j be arbitrary positive integers. In Sect. 2, we saw that P is non-conglomerable in the following manner:

> **Fact**$_1$. $1 \geq P(A|V_i) \geq 1$ and $0 \geq P(A|H_j) \geq 0$, yet $P(A) < 1$ or $P(A) > 0$.[38]

Since $P(q_{1,1}|q_{1,1}) = 1$ and $P(\neg q_{1,1}|q_{1,1}) = 0$, **Fact**$_1$ entails:

> **Fact**$_2$. $P(q_{1,1}|q_{1,1}) \geq P(A|V_i) \geq P(q_{1,1}|q_{1,1})$ and $P(\neg q_{1,1}|q_{1,1}) \geq P(A|H_j) \geq P(\neg q_{1,1}|q_{1,1})$, yet $P(A|\Omega) < P(q_{1,1}|q_{1,1})$ or $P(A|\Omega) > P(\neg q_{1,1}|q_{1,1})$.

Hence, we might expect that P can almost represent some rational comparative confidence relation $\succeq$ that is non-conglomerable in an analogous manner:

> **Condition.** $q_{1,1}|q_{1,1} \succeq A|V_i \succeq q_{1,1}|q_{1,1}$ and $\neg q_{1,1}|q_{1,1} \succeq A|H_j \succeq \neg q_{1,1}|q_{1,1}$, yet $q_{1,1}|q_{1,1} \succ A|\Omega$ or $A|\Omega \succ \neg q_{1,1}|q_{1,1}$.

As it turns out, however, there is no rational comparative confidence relation that satisfies this condition.[39] So, if P can almost represent some rational comparative confidence relation $\succeq$ that is non-conglomerable, then $\succeq$ must be non-conglomerable in a manner not entirely analogous to that in which P is non-conglomerable. Explicit construction—of the sort I provide in Sect. 4—is required to see that there is indeed such a relation.

In general, then, one cannot simply "read off" a rational comparative confidence relation from a non-conglomerable probability function and infer that the former will be non-conglomerable. Qualitative non-conglomerability must be investigated on its own terms.

[38] Although I only explicitly appealed to the fact that $P(A) \neq 0$ or $P(A) \neq 1$ to show that P is non-conglomerable, the assumption that A is in the algebra on which P is defined ensures that $P(A) < 1$ or $P(A) > 0$.

[39] I show here that $A|V_i \not\succeq q_{1,1}|q_{1,1}$ for any rational comparative confidence relation $\succeq$. First, by **Verified Contingency**, $q_{1,1}|q_{1,1} \succeq V_i|V_i$. Next, note that $(A \cap V_i)$ is a proper subset of $(V_i \cap V_i)$. So, by the aforementioned Theorem 3 in Koopman (1940a), $V_i|V_i \succ A|V_i$. Thus, by **Transitivity**, $q_{1,1}|q_{1,1} \succ A|V_i$. Hence, $A|V_i \not\succeq q_{1,1}|q_{1,1}$.

5.2 Reason 2: The qualitative paradox entails the quantitative paradox

In the previous section, I showed that the quantitative non-conglomerability result of Sect. 2 does not, in any straightforward sense, entail the qualitative paradox. By contrast, I now show that there is a sense in which the qualitative non-conglomerability result of Sect. 4 does entail the quantitative paradox. Hence, there is a sense in which the qualitative paradox entails the quantitative paradox, although the converse is not the case.

Let P be an arbitrary probability function that can almost represent the comparative confidence relation $\succeq$ of Sect. 4. As before, let $\pi_1 = \{V_i : i \in \mathbb{N}\}$ and $\pi_2 = \{H_j : j \in \mathbb{N}\}$. Then, by Lemma 3, Lemma 4, and the fact that $\succeq$ is almost representable by P:

- $P(q_{1,1}|q_{1,1}) \geq P(A|v) \geq P[q_{1,1}|(q_{1,1} \cup q_{1,2})]$ for all $v \in \pi_1$, and
- $P[q_{1,1}|(q_{1,1} \cup q_{1,2} \cup q_{1,3})] \geq P(A|h) \geq P(\neg q_{1,1}|q_{1,1})$ for all $h \in \pi_2$.

Note that $P(q_{1,1}|q_{1,1}) = 1$ and, by finite additivity, $P(\neg q_{1,1}|q_{1,1}) = 0$. Additionally, $P[q_{1,1}|(q_{1,1} \cup q_{1,2})] = \frac{1}{2}$ and $P[q_{1,1}|(q_{1,1} \cup q_{1,2} \cup q_{1,3})] = \frac{1}{3}$.[40] Plugging these values into the above then yields:

- $1 \geq P(A|v) \geq \frac{1}{2}$ for all $v \in \pi_1$, and
- $\frac{1}{3} \geq P(A|h) \geq 0$ for all $h \in \pi_2$.

Since it cannot be that both $P(A) \geq \frac{1}{2}$ and $P(A) \leq \frac{1}{3}$, P must be non-conglomerable in at least one of π_1 and π_2. Hence, P is non-conglomerable. Since P was arbitrary, it follows that any probability function that can almost represent $\succeq$ is non-conglomerable. Finally, since $\succeq$ is indeed almost representable by some probability function—for example, by de Finetti's—it follows that there is some probability function that is non-conglomerable. In this sense, the qualitative paradox entails the quantitative paradox.

Although the fact that the qualitative paradox entails the quantitative paradox (but not vice versa) is noteworthy in its own right, it acquires special

[40] By Lemma 2, $\{q_{1,1}, q_{1,2}\}$ and $\{q_{1,1}, q_{1,2}, q_{1,3}\}$ are a 2-scale and a 3-scale, respectively. So, $q_{1,1}|(q_{1,1} \cup q_{1,2}) \approx q_{1,2}|(q_{1,1} \cup q_{1,2})$ and $q_{1,1}|(q_{1,1} \cup q_{1,2} \cup q_{1,3}) \approx q_{1,2}|(q_{1,1} \cup q_{1,2} \cup q_{1,3}) \approx q_{1,3}|(q_{1,1} \cup q_{1,2} \cup q_{1,3})$. Additionally, by finite additivity and the fact that $\succeq$ is almost representable by P, it follows that $P[q_{1,1}|(q_{1,1} \cup q_{1,2})] = P[q_{1,2}|(q_{1,1} \cup q_{1,2})] = \frac{1}{2}$ and $P[q_{1,1}|(q_{1,1} \cup q_{1,2} \cup q_{1,3})] = P[q_{1,2}|(q_{1,1} \cup q_{1,2} \cup q_{1,3})] = P[q_{1,3}|(q_{1,1} \cup q_{1,2} \cup q_{1,3})] = \frac{1}{3}$.

significance if one regards comparative confidence as somehow more fundamental than credence. In particular, if a rational agent's credences are merely an (almost) representation of her attitudes of comparative confidence, then the qualitative paradox explains why the quantitative paradox arises at all: the latter must—out of representational necessity—arise.

5.3 Reason 3: Relevance to infinitesimals

In this section, I show that the qualitative paradox entails that any probability function—whether real-valued or extended-valued—that treats some countably infinite lottery as fair must be non-conglomerable. Hence, one cannot avoid the *quantitative* paradox of non-conglomerability merely by appealing to infinitesimals.

First, as I showed in the previous section, any probability function that can almost represent the non-conglomerable comparative confidence relation $\succeq$ of Sect. 4 must be non-conglomerable. Additionally, as I remarked in Sect. 4.2, $\succeq$ need not satisfy **Independence** in order to be non-conglomerable; it need only satisfy Koopman's axioms and **Fairness**. Hence, a general result holds: any probability function that can almost represent a comparative confidence relation that satisfies Koopman's axioms and treats some countably infinite lottery as fair must be non-conglomerable.

Next, let L be a countably infinite lottery, and let L be fair with respect to some probability function P. That is, let $P(i) = P(j)$ for any possible outcomes i, j of L. In the Appendix, I show that any (conditional) probability function can almost represent some comparative confidence relation that satisfies Koopman's axioms. So, if the only constraints on some comparative confidence relation $\succeq$ are that it satisfies Koopman's axioms and treats L as fair, then clearly P can almost represent $\succeq$. Because the qualitative paradox ensures that $\succeq$ is non-conglomerable, it follows that P is non-conglomerable as well. Thus, any probability function that treats some countably infinite lottery as fair must be non-conglomerable.

In Sect. 2, we saw that de Finetti's real-valued probability function treated two countably infinite lotteries as fair and thereby assigned each possible outcome probability 0. By contrast, some have argued that it is more appropriate to characterize a probabilistically fair, countably infinite lottery via a probability function that can take on values beyond the real numbers. For example, Wenmackers and Horsten (2013) argue that the possible outcomes of such a lottery should be assigned a positive yet infinitesimal value.

By the result of the previous paragraph, it follows that any such probability function must be non-conglomerable.[41]

A consequence of this fact is that would be inappropriate to criticize a purported characterization of a countably infinite, probabilistically fair lottery solely on the grounds that it fails to be conglomerable.[42] The qualitative paradox ensures that *any* such characterization is bound to be non-conglomerable. One cannot escape non-conglomerability merely by expanding the range of one's probability function.

6 Responses to the qualitative paradox

A number of responses to the qualitative paradox of non-conglomerability are available. Although I will not endorse any particular response, I will canvass some possible options in this section. In Sect. 6.1, I discuss the option of accepting the paradox. In Sects. 6.2–6.4, I discuss options for denying it.

6.1 Option 1: Accept the paradox

Recall that the qualitative paradox is a consequence of Koopman's axioms and the assumption that a rational agent can treat a countably infinite lottery as qualitatively fair.[43] One response to the paradox is simply to bite the bullet: accept Koopman's axioms, accept the assumption about qualitative fairness, and thereby accept the paradox as a counterintuitive fact about rationality.

de Finetti (1972) himself was a notable defender of the quantitative paradox of non-conglomerability. He had little difficulty stomaching it:

> At worst, one could experience that uneasy feeling which in all fields of mathematics is caused by the introduction of the new characteristic properties of infinity, and which lasts until it is relieved by familiarity and reflection. (p. 104)

[41] Pruss (2014) already observes that any such probability function is non-conglomerable. However, while his observation draws on specific mathematical properties of such functions, the route provided here illustrates the representational necessity of this fact.

[42] Pruss (2014) levels this criticism, among others.

[43] As I remarked in Sect. 4.2, **Independence** is not necessary to generate the paradox.

Perhaps we should heed de Finetti's words and simply learn to live with qualitative non-conglomerability as well.[44]

6.2 Option 2: Deny some of Koopman's axioms

Alternatively, we might deny the qualitative paradox. One way to do so is by deying some of Koopman's axioms that are appealed to in the argument for the paradox.[45] However, it is unclear which axioms are most to "blame" for the paradox—**Antisymmetry**? **Composition**?—or whether there is positive reason to reject any of them. Moreover, although some authors—including Luce (1968) and Krantz et al. (1971)—have argued that Koopman's axiomatization is somewhat inelegant, there is widespread agreement that Koopman's axioms are plausible at least as rational *constraints* on comparative conditional confidence. So, it is unclear how promising this option is.

That said, alleged counterexamples have been raised to some widely accepted axioms of rational comparative *un*conditional confidence that are analogous to some of Koopman's axioms (e.g., **Transitivity**).[46] However, the question of whether any such alleged counterexamples are plausible—or can be reformulated as plausible counterexamples to any of Koopman's axioms—lies beyond the scope of the present paper.

6.3 Option 3: Deny Fairness

Another way to block the qualitative paradox is to deny that any rational comparative confidence relation can satisfy **Fairness**. That is, we might reject the rational permissibility of any comparative confidence relation that treats some countably infinite lottery as fair. Because **Fairness** appears to be logically consistent with Koopman's axioms, pursuing this route would likely involve appealing to principles of rational comparative confidence that go beyond Koopman's axioms. What might such principles look like?

Here is an obvious candidate principle:

[44] Additional authors who accept the quantitative paradox include Hill (1980), Kadane et al. (1986), and Arntzenius et al. (2004).

[45] As I said in footnote 24, the argument appeals to all of Koopman's axioms except for **Alternative Presumption.**

[46] See Fishburn (1986) for discussion.

> **Conglomerability.** Every rational comparative confidence relation is conglomerable.

Clearly, **Conglomerability** suffices to block the satisfaction of **Fairness** by any comparative confidence relation that satisfies Koopman's axioms. But should **Conglomerability** be accepted as a principle of rationality?

If **Conglomerability** is accepted as a *basic* principle of rationality—one that does not follow from other principles of rationality—then it has the potential to seem ad hoc. A non-conglomerable comparative confidence relation is counterintuitive, to be sure, but is that reason enough to accept **Conglomerability** as a basic principle of rationality? Perhaps we should follow de Finetti's example and simply accept qualitative non-conglomerability as yet another counterintuitive phenomenon that arises from considerations involving infinity. Alternatively, if **Conglomerability** is accepted as a *derived* principle of rationality—one that does follow from other principles of rationality—then the question arises as to which principles of rationality it follows from. But the nature of such principles is not immediately clear.

Another route takes its cue from de Finetti's original probabilistic setup. Some have argued that no agent should have a *probabilistically* fair credence function in the outcomes of a countably infinite lottery on the grounds that any such function violates countable additivity.[47] Countable additivity is the countable extension of finite additivity:

> (Countable additivity.) For mutually disjoint propositions $A_1, A_2, \ldots$,
>
> $$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots \tag{31}$$

It is easy to see that de Finetti's probability function P is not countably additive.[48] Thus, insofar as countable additivity is a rational constraint on credence—about which more soon—P is not a rationally permissible credence function. Moreover, Hill & Lane (1986) show that a probability function is countably additive if and only if it is conglomerable in all countable partitions. Hence, to ensure qualitative conglomerability (at least, in all countable partitions), we might search for some constraint on comparative confidence that (1) is a qualitative counterpart of countable additivity and (2) conflicts with **Fairness**.

[47] See Bartha (2004) for discussion.

[48] Because $P(V_i) = 0$ for every positive integer i, $P(V_1) + P(V_2) + \ldots = 0 + 0 + \ldots = 0$. However, $P(\Omega) = P(V_1 \cup V_2 \cup \ldots) = 1$. So, $P(V_1 \cup V_2 \cup \ldots) \neq P(V_1) + P(V_2) + \ldots$

As it turns out, there is indeed such a constraint. Villegas (1964) shows that, for an unconditional comparative confidence relation $\succeq$ that satisfies constraints analogous to Koopman's axioms, a necessary—but not sufficient—condition for $\succeq$ to be representable by a countably additive probability function is that $\succeq$ satisfies following:

> **Monotone Continuity.**
> Suppose:
>
> (a) $(A_1, A_2, \ldots)$ is a monotone non-decreasing sequence of propositions.[49] That is: $A_1 \subseteq A_2 \subseteq \ldots$
>
> (b) $A = \cup_i A_i$.
>
> (c) $B \succeq A_i$ for all i.
>
> Then: $B \succeq A$.[50]

In words, **Monotone Contintuity** says: if the monotone non-decreasing propositions $A_1, A_2, \ldots$ converge to proposition A, then the judgment that one should be at least as confident in B as in each A_i carries through in the limit.[51]

Fairness, in conjunction with Koopman's axioms, is inconsistent with **Monotone Continuity**. To see this, let $\succeq$ be the non-conglomerable comparative confidence relation of Sect. 4.[52] Also, let $A_1 = V_1$, $A_{i+1} = (A_i \cup V_{i+1})$ for $i > 1$, $A = \cup_i A_i = \Omega$, and $B = \Omega - A_1$. It follows that $A_1 \subseteq A_2 \subseteq \ldots$ and

[49] $A, B, A_1, A_2, \ldots$ are arbitrary propositions in the algebra $\mathcal{F}$ on which $\succeq$ is defined. I follow the formulation of Fishburn (1986, p. 342), who allows $\mathcal{F}$ to be an arbitrary Boolean algebra (not necessarily a σ-algebra, as in the original formulation of Villegas).

[50] Chateauneuf & Jaffray (1984) show that a necessary *and* sufficient condition for $\succeq$ to be representable by a countably additive probability function is that $\succeq$ satisfies **Monotone Continuity** as well as a particular "Archimedean" condition. At the time of this writing, it does not appear to be known what necessary and sufficient conditions may be given under which $\succeq$ is almost representable by a countably additive probability function. However, see Chuaqui & Malitz (1983) and Schwarze (1989) for results—all of which involve **Monotone Continuity**—in this direction.

[51] I adapt the paraphrase of Fishburn (1986, pp. 342–343) here.

[52] In what follows, I will take '$C \succeq D$' to mean that $C|\Omega \succeq D|\Omega$.

$B \succeq A_i$ for all i.[53] Nonetheless, $B \not\succeq A$.[54] So, $\succeq$ violates **Monotone Continuity**. Thus, insofar as we should accept Koopman's axioms and **Monotone Continuity**, we should reject **Fairness**—and the rational permissibility of the comparative confidence relation in question.

Now, this is not the place to assess whether countable additivity—or its qualitative counterpart of **Monotone Continuity**—is indeed a rational constraint on doxastic attitudes. That said, although there has been considerable discussion concerning the rational merits of countable additivity,[55] there has been correspondingly little discussion concerning that of **Monotone Continuity**.[56] Moreover, because **Monotone Continuity** is only a necessary—but not sufficient—condition for a rational comparative confidence relation to be representable by a countably additive probability function, arguments for or against countable additivity do not necessarily translate into arguments for or against **Monotone Continuity**. Nonetheless, analogous questions arise:

- Is **Monotone Continuity** more intuitively plausible than its negation?[57]

- Does violating **Monotone Continuity** make one susceptible to a

[53]First, $A_1 \subseteq A_2 \subseteq \ldots$ because $V_1 \subseteq (V_1 \cup V_2) \subseteq \ldots$ Next, let i, j be arbitrary positive integers. By **Fairness**, $V_i|\Omega \approx V_j|\Omega$. So, $V_1|\Omega \approx V_2|\Omega, \ldots, V_{i+1}|\Omega \approx V_{i+2}|\Omega$. Next, by a straightforward application of Theorem 6 of Koopman (1940a), it follows that $(V_1 \cup \ldots \cup V_{i+1})|\Omega \approx (V_2 \cup \ldots \cup V_{i+2})|\Omega$. Additionally, note that $B = \Omega - A_1 = \Omega - V_1 = (V_2 \cup V_3 \cup \ldots)$. So, $(V_2 \cup \ldots \cup V_{i+2}) \subseteq B$. Hence, by the aforementioned Theorem 3 of Koopman (1940a), $B|\Omega \succeq (V_2 \cup \ldots \cup V_{i+2})|\Omega$. Finally, since $A_{i+1} = (V_1 \cup \ldots \cup V_{i+1})$, $A_{i+1}|\Omega \approx (V_2 \cup \ldots \cup V_{i+2})|\Omega$. By **Transitivity**, $B|\Omega \succeq A_{i+1}|\Omega$. For similar reasons, $B|\Omega \succeq A_1|\Omega$. Hence, $B|\Omega \succeq A_i|\Omega$. That is, $B \succeq A_i$.

[54]Note that, since $V_1 \cap V_i = \emptyset$ for all $i > 1$, $B = (V_2 \cup V_3 \cup \ldots)$ is a proper subset of $A = (V_1 \cup V_2 \cup \ldots)$. So, the aforementioned Theorem 3 of Koopman (1940a) entails that $A|\Omega \succ B|\Omega$. Hence, $B \not\succeq A$.

[55]See de Finetti (1972), Bartha (2004), Howson (2008), and Easwaran (2013b).

[56]For example, Fishburn (1986), in a comprehensive literature review on axioms of comparative probability, merely remarks that "Monotone continuity is quite appealing" (343). Similarly, Fine (1973) only says, "Unlike [other axioms of comparative probability], we are inclined to view [monotone continuity] as attractive but not necessary for a characterization of [comparative probability]" (21). However, both authors discuss relevant representation theorems at length.

[57]See de Finetti (1972, pp. 91–2), who argues that the intuitive plausibility of the possibility of a *probabilistically* fair, countably infinite lottery is a reason to reject countable additivity.

"qualitative" Dutch Book or other pragmatic failing?[58]

- Can **Monotone Continuity** be justified on epistemic grounds?[59]
- And so on.[60]

I cannot address these questions here, but they are the sorts of questions that should be asked in assessing whether the non-conglomerable comparative confidence relation of Sect. 4 is rationally permissible.

6.4 Option 4: Deny the legitimacy of $\mathcal{F}$

The qualitative non-conglomerability result of Sect. 4 only has rational significance if all of the propositions in $\mathcal{F}$ can figure in some rational agent's comparative confidence relation. There are a couple of ways one might deny this to be the case and thereby block the qualitative paradox.

First, one might deny that some propositions in $\mathcal{F}$ are entertainable by any agent (rational or otherwise). In particular, one might deny that $A = [q_{1,1} \cup (q_{1,2} \cup q_{2,2}) \cup (q_{1,3} \cup q_{2,3} \cup q_{3,3}) \cup \ldots]$ is entertainable because it is an infinite set of propositions. If A is not entertainable by any agent, then clearly the argument for the qualitative paradox—which appeals essentially to comparisons of conditional confidence that involve A—does not go through.

Recall, however, that A is merely the proposition that the number drawn by L_2 is greater than or equal to that drawn by L_1. Why couldn't one entertain this proposition? (And did I not just do so?) More generally, since $\mathcal{F}$ only contains propositions that are formed by Boolean operations (with respect to Ω) on quite ordinary and non-gerrymandered propositions—namely, every V_i, every H_j, and A—it seems implausible that some propositions in $\mathcal{F}$ should fail to be entertainable.

Alternatively, one might allow that every proposition in $\mathcal{F}$ is entertainable yet hold that some propositions in $\mathcal{F}$ cannot figure in any rational agent's

[58]See Icard (2016), who provides a pragmatic argument for other constraints on comparative confidence.

[59]See Fitelson & McCarthy (2014), who generalize Joyce (1998, 2009)'s epistemic arguments for probabilism to justify other constraints on comparative confidence.

[60]For example, Easwaran (2013b, Sect. 2) provides an argument for countable additivity that is neither "pragmatic" nor "epistemic". Can Easwaran's argument can be reformulated as an argument for **Monotone Continuity**?

attitudes of comparative confidence. In particular, one might adopt a qualitative analogue of the probabilistic finitism espoused by Jaynes (2003, pp. 43–44).[61] According to Jaynes' "finite sets policy", we should only ascribe probabilities directly to finite sets of propositions; infinite sets, like A, have probabilities only relative to particular limiting processes on finite sets.[62] Jaynes argues that adoption of his "finite sets policy" is key to blocking a number of probabilistic paradoxes—including that of non-conglomerability (*ibid.*, Ch. 15). One might adopt an analogous policy towards rational comparative confidence and thereby attempt to block the qualitative paradox of non-conglomerability.

Whatever the merits of the "finite sets policy" with respect to quantitative probability, however, an analogous policy does not seem plausible with respect to rational comparative confidence. For example, if one can entertain the proposition A—and I just argued that one easily can—then it seems clear that one should be at least as confident in A as in any of its subsets. Limiting processes on finite sets of propositions seem wholly irrelevant to the question of the rationality of this judgment. More generally, Koopman's axioms—all of which involve simple set-theoretic relations (if any) among propositions—seem just as intuitively plausible with respect to infinite sets of propositions as with respect to finite sets. Thus, if one wishes to block the qualitative paradox of non-conglomerability via Jaynes-style finitism, then it seems one must deny that some of Koopman's axioms apply to infinite sets. However, unless one adopts a more radical finitism about the possible contents of thought, it is unclear why this should be the case.

[61]Although Jaynes never uses the term 'probabilistic finitism', it is clear that he is some kind of finitist about probability: "In our view, an infinite set cannot be said to possess any 'existence' and mathematical properties at all—at least, in probability theory—until we have specified the limiting process that is to generate it from a finite set. In other words, we sail under the banner of Gauss, Kronecker, and Poincaré rather than Cantor, Hilbert, and Bourbaki." (*ibid.*, xxii)

[62]More precisely, let X be a countably infinite set of propositions, $a = (a_1, a_2, \ldots)$ a countably infinite sequence of finite sets of propositions, and P a probability function on the members of a. Then, X has probability p relative to a just in case $X = \bigcup_n a_n$ and $p = \lim_{n\to\infty} P(a_n)$. Similarly for uncountably infinite sets.

7 Conclusion

Since de Finetti's discovery of non-conglomerable probability functions, non-conglomerability has been a lively topic of investigation in the philosophical and statistical literature. In this paper, I have shown that the phenomenon of non-conglomerability—along with its attendant host of philosophical issues—naturally extends to comparative confidence. I close with some open technical questions.

In Sect. 4, I showed that de Finetti's probability function—which is finitely additive but not countably additive—can be reformulated as a rational comparative confidence relation that is non-conglomerable. However, it remains to be seen whether, and how, other examples of non-conglomerable probability functions can be qualitatively reformulated. For example, does the probability function of Kadane et al. (1986, Sect. 6)—which is countably additive yet non-conglomerable in some uncountable partition—correspond to a rational comparative confidence relation that is non-conglomerable? More generally, can necessary and sufficient conditions be given under which a rational comparative confidence relation is conglomerable in all partitions?

As I said in Sect. 6.3, a probability function is conglomerable in all countable partitions if and only if it is countably additive. That said, the aforementioned example of Kadane et al. (1986) shows that countable additivity alone is not sufficient to ensure probabilistic conglomerability in all partitions. Thus, to identify necessary and sufficient conditions for qualitative conglomerability in all partitions, it seems reasonable to seek a constraint on comparative confidence that (1) is a qualitative analogue of countable additivity but (2) is a stronger constraint on comparative confidence than countable additivity is on probability.

As in Sect. 6.3, **Monotone Continuity** seems to fit the bill yet again. We saw there that **Monotone Continuity** is a qualitative analogue of countable additivity. Unlike countable additivity, however, **Monotone Continuity** does not apply only to countably infinite sets of propositions; it holds with respect to *any* monotone non-decreasing sequence of propositions. In this regard, **Monotone Continuity** is indeed a stronger constraint on comparative confidence than countable additivity is on probability. Thus, the foregoing considerations suggest the following conjecture:

> **Conjecture.** Let $\succeq$ be a comparative confidence relation that satisfies Koopman's axioms. Then, $\succeq$ is conglomerable in all par-

> titions if and only if $\succeq$ satisfies **Monotone Continuity** with respect to all monotone non-decreasing sequences of propositions.[63]

I leave this conjecture, as well as the previous questions, open for future work.

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Appendix

In this Appendix, I show that the non-conglomerable comparative confidence relation $\succeq$ of Sect. 4 is almost representable by de Finetti's non-conglomerable probability function P of Sect. 2. That is, I show:

> **Almost Representability.** If $A|B \succeq C|D$, then $P(A|B) \geq P(C|D)$.

Note that the only constraints on $\succeq$ are Koopman's axioms, **Fairness**, and **Independence**. So, we must show that P satisfies suitable probabilistic analogues of them. For example, since $k|k \succeq a|h$ by **Verified Contingency**, we must show that $P(k|k) \geq P(a|h)$. Similarly, since $V_i|\Omega \approx V_k|\Omega$ by **Fairness**, we must show that $P(V_i|\Omega) = P(V_k|\Omega)$—i.e., that $P(V_i) = P(V_k)$.

I begin with Koopman's axioms. Because I will not yet appeal to any properties of P beyond its satisfying constraints (1)–(4) from Sect. 2—I will not even appeal to the fact that P is real-valued—it will follow that any conditional probability function satisfying those conditions also satisfies the relevant probabilistic analogues of Koopman's axioms. Thus, any conditional probability function—real-valued or extended-valued—can almost represent some comparative confidence relation that satisfies Koopman's axioms.

[63]This conjecture is further suggested by the result of Seidenfeld et al. (2013), who generalize the notion of countable additivity to κ-additivity for any infinite cardinality κ. They show that, if a probability function P fails to be κ-additive for some uncountable κ (but satisfies particular structural constraints), then P is non-conglomerable in a partition of cardinality κ. Hence, insofar as **Monotone Continuity** with respect to κ-sized sequences of monotone non-decreasing propositions is a qualitative counterpart of κ-additivity—another open question in itself—some form of this conjecture is all the more reasonable.

1. **Verified Contingency.** Recall that $P[(A \cap B)|B] = P(A|B)$ (cf. footnote 14). So, $P(a|h) = P[(a \cap h)|h]$ and $P(\neg a|h) = P[(\neg a \cap h)|h]$. By finite additivity and constraint (2), $P(a|h) + P(\neg a|h) = P[(a \cap h)|h] + P[(\neg a \cap h)|h] = P(h|h) = 1$. Since $P(\neg a|h) \geq 0$, it follows that $P(a|h) \leq 1$. Thus, since $P(k|k) = 1$, $P(k|k) \geq P(a|h)$.

2. **Implication.** Suppose $a|h \succeq k|k$, so that $h \subseteq a$. Then $a = (h \cup h')$ for some $h' \in \mathcal{F}$ such that $h \cap h' = \emptyset$. So, by finite additivity, $P(a|h) = P[(h \cup h')|h] = P(h|h) + P(h'|h) = 1 + P(h'|h) \geq 1 = P(k|k)$, as desired.

3. **Reflexivity.** Obviously, $P(a|h) \geq P(a|h)$.

4. **Transitivity.** Suppose that $P(c|l) \geq P(b|k)$ and $P(b|k) \geq P(a|h)$. Obviously, $P(c|l) \geq P(a|h)$.

5. **Antisymmetry.** Suppose that $P(b|k) \geq P(a|h)$. Then, as above, $P(b|k)+P(\neg b|k) = P[(b\cap k)|k]+P[(\neg b\cap k)|k] = P(k|k) = 1$. Similarly, $P(a|h) + P(\neg a|h) = 1$. Thus, $P(\neg a|h) = 1 - P(a|h) \geq 1 - P(b|k) = P(\neg b|k)$, as desired.

6. **Composition.**
   Suppose:

   (a) $(a_1 \cap b_1) \cap h_1 \neq \emptyset$ and $(a_2 \cap b_2) \cap h_2 \neq \emptyset$.

   (b) $P(a_2|h_2) \geq P(a_1|h_1)$.

   (c) $P[b_2|(a_2 \cap h_2)] \geq P[b_1|(a_1 \cap h_2)]$.

   By constraint (4), $P[(a_2\cap b_2)|h_2] = P[b_2|(a_2\cap h_2)]P(a_2|h_2)$ and $P[(a_1 \cap b_1)|h_1] = P[b_1|(a_1\cap h_1)]P(a_1|h_1)$. Thus, by (b) and (c), $P[(a_2\cap b_2)|h_2] \geq P[(a_1 \cap b_1)|h_1]$.

7. **Decomposition.**
   Suppose:

   (a) $(a_1 \cap b_1) \cap h_1 \neq \emptyset$ and $(a_2 \cap b_2) \cap h_2 \neq \emptyset$.

   (b) $P(a_1|h_1) \geq P(a_2|h_2)$.

   (c) $P[(a_2 \cap b_2)|h_2] \geq P[(a_1 \cap b_1)|h_1]$.

As above, $P[(a_2 \cap b_2)|h_2] = P[b_2|(a_2 \cap h_2)]P(a_2|h_2)$ and $P[(a_1 \cap b_1)|h_1] = P[b_1|(a_1 \cap h_1)]P(a_1|h_1)$. Thus, by (b) and (c), it follows that $P[b_2|(a_2 \cap h_2)] \geq P[b_1|(a_1 \cap h_1)]$.

8. **Alternative Presumption.** Suppose $P(r|s) \geq P[a|(b \cap h)]$ and $P(r|s) \geq P[a|(\neg b \cap h)]$. By constraint (4), $P[(a \cap b)|h] = P[a|(b \cap h)]P(b|h)$ and $P[(a \cap \neg b)|h] = P[a|(\neg b \cap h)]P(\neg b|h)$. So, by finite additivity, $P[(a \cap b)|h] + P[(a \cap \neg b)|h] = P(a|h)$. Using finite additivity again,

$$\begin{aligned} P(a|h) &= P[a|(b \cap h)]P(b|h) + P[a|(\neg b \cap h)]P(\neg b|h) && (32) \\ &= P[a|(b \cap h)]P(b|h) + P[a|(\neg b \cap h)][1 - P(b|h)] && (33) \\ &\leq P(r|s)P(b|h) + P(r|s)[1 - P(b|h)] && (34) \\ &= P(r|s), && (35) \end{aligned}$$

as desired.

9. **Subdivision.**
Suppose the propositions $a_1, \ldots, a_n, b_1, \ldots, b_n$ are such that:

   (a) $a_i \cap a_j = b_i \cap b_j = \emptyset$ for all $i, j = 1, \ldots, n$ such that $i \neq j$.

   (b) $a = (a_1 \cup \ldots \cup a_n) \neq \emptyset$ and $b = (b_1 \cup \ldots \cup b_n) \neq \emptyset$.

   (c) $P(a_n|a) \geq \ldots \geq P(a_1|a)$ and $P(b_n) \geq \ldots \geq P(b_1|b)$.

   Since finite additivity entails that $P(a_1|a) + \ldots + P(a_n|a) = P(a|a) = 1$, it follows from (c) that $P(a_1|a) \leq \frac{1}{n}$. For the same reason, $P(b_1|b) \leq \frac{1}{n}$. Since $P(b_n|b) \geq P(b_1|b)$, it follows that $P(b_n|b) \geq \frac{1}{n} \geq P(a_1|a)$.

Fairness and **Independence** place stricter requirements on almost representability than Koopman's axioms in that not every conditional probability function satisfies their probabilistic analogues. Nonetheless, it is easy to see that P does. In particular, by definition of P:

- **Probabilistic Fairness.** For every $i, j \in \mathbb{N}$: $P(V_i) = P(V_j)$ and $P(H_i) = P(H_j)$.
- **Probabilistic Independence.** For every $i, j \in \mathbb{N}$: $P(H_j|V_i) = P(H_j)$ and $P(V_i|H_j) = P(V_i)$.

Since P satisfies probabilistic analogues of Koopman's axioms, **Fairness**, and **Independence**, it follows that $\succeq$ is almost representable by P.

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